that the lines parallel to the plane of vibration were distinctly sharper than those at right angles to the plane.

2. In this paper Professor Stromquist derives certain statistical formulas, in particular those for the standard deviation and the coefficient of correlation, that result from a given correlation, or double entry, table by adding new individuals to the table, by translating individuals in the table, and by superimposing one correlation table upon another.

3. Mr. Fitch discussed the trajectories due to a flow of water from an orifice subject to constant angle and constant kinetic energy.

4. Professor Light gave the geometrical conditions that must be fulfilled when the extraneous loci are cusp-loci, tac-loci, and singular solutions.

5. Professor Sisam discussed some properties of algebraic correspondences between two given algebraic curves of which at least one is rational.

6. Mr. Kitchen brought out, among other good points, the fact that high school students do not know how to draw conclusions from definite statements.

7. Experimental work on the rate of thermal expansion of glass from room temperature to 750 degrees Centigrade has brought out the relation between this and other thermal properties. Professor Pietenpol showed how the expansion of glass is of particular importance in its relation to the annealing of glass, and that a determination of the rate of expansion at high temperatures may be used as an exact method of determining the suitable annealing temperature.

8. Mr. McNatt took up the derivation of the equation of the catenary, and some of the properties of the equation were applied to the solutions of problems arising in connection with cables used in mines.

9. Professor Sperry gave a proof of a well known theorem that the average value of the ratio of the weight of the observed value of an unknown to its adjusted value for a series of unknowns is equal to the number of unknowns divided by the number of observations. Instead of the usual proof by undetermined coefficients, certain transformations were effected by means of determinants. This proof is believed to be superior in directness and simplicity.

G. H. Light, *Secretary-Treasurer.*

RATIONAL TRIANGLES AND QUADRILATERALS.

By L. E. Dickson, University of Chicago.

1. **The questions treated.** The chief object of this paper is to make a material simplification in Kummer's classic investigation of rational quadrilaterals. Incidentally it is shown that every rational triangle may be formed by juxtaposing two rational right triangles, so that it suffices to know Diophantus's complete solution of $x^2 + y^2 = z^2$ in rational numbers. From the latter will be deduced all solutions in integers, a problem usually treated independently of the former problem of the rational solutions. For most equations the two problems are essentially distinct.

2. **Rational solutions of $x^2 + y^2 = z^2$.** Diophantus (*Arithmetica*, II, 8) took in effect $y = x(m/n) - z$, where m/n is a fraction in its lowest terms [the value of $(y + z)/x$]. Thus

$$x = \frac{2mnz}{m^2 + n^2}, \qquad y = \frac{xm}{n} - z = \frac{(m^2 - n^2)z}{m^2 + n^2}.$$

Hence *the sides of any rational right triangle are proportional to*

$$(1) \qquad 2mn, \qquad m^2 - n^2, \qquad m^2 + n^2,$$

where m and n are relatively prime positive integers.[1] Diophantus spoke of the right triangle with the sides (1) as that *formed from m, n.*

3. **Integral solutions of $x^2 + y^2 = z^2$.** We shall prove that *all positive integral solutions of $x^2 + y^2 = z^2$ are given without duplication by*

$$(2) \qquad x = 2mnl, \qquad y = (m^2 - n^2)l, \qquad z = (m^2 + n^2)l, \qquad m > n > 0,$$

where m and n are relatively prime integers and not both odd, while l is a positive integer.[2]

When m and n are both odd, the numbers (1) rearranged are the doubles of the numbers $\frac{1}{2}(m^2 - n^2)$, mn, $\frac{1}{2}(m^2 + n^2)$, which are the sides of the right triangle formed from the integers $\frac{1}{2}(m + n)$, $\frac{1}{2}(m - n)$. Hence we may restrict attention to the case in which m and n are relatively prime and not both odd, so that one is even and the other odd. Thus the last two numbers (1) are odd; any common divisor of those two would divide their sum $2m^2$ and their difference $2n^2$, and hence divide m^2 and n^2, which are relatively prime. Hence the last two numbers (1) have no common divisor > 1. Thus if their products by the same irreducible fraction a/b are both integers, they must be divisible by b, whence $b = 1$. Hence integers are the only rational numbers whose products by all the numbers (1) give integers, when m and n are relatively prime and not both odd. By Diophantus's result in § 2, all integral solutions (like all rational solutions) are products of the numbers (1) by rational numbers. To complete the proof of our present theorem, it remains only to show that the integral solutions (2) contain no duplicates.

Suppose that the numbers (2) coincide with

$$2MNL, \qquad (M^2 - N^2)L, \qquad (M^2 + N^2)L,$$

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[1] To give another interesting proof, we note that, if θ is an angle of any rational right triangle, $\sin\theta$ and $\cos\theta$ are rational and hence $t = \tan\frac{1}{2}\theta = \sin\theta/(1 + \cos\theta)$ is rational. Conversely, if t is rational, also

$$\sin\theta = \frac{2t}{1 + t^2}, \qquad \cos\theta = \frac{1 - t^2}{1 + t^2}$$

are rational. The first mathematical article in this MONTHLY [*1894*, 6–11] was one on this subject by the present writer, who was co-editor of the MONTHLY from 1902 to 1908.

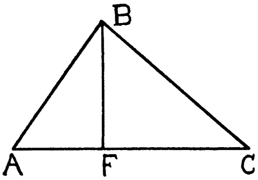
[2] The proof by Kronecker, *Vorlesungen über Zahlentheorie*, 1901, p. 35, is open to the serious objection raised by the writer in "Fallacies and misconceptions in Diophantine analysis," *Bulletin of the American Mathematical Society*, April, 1921. Moreover, it is not proved that l is an integer.

where M and N are relatively prime positive integers. By division,

$$\frac{m^2 \mp n^2}{2mn} = \frac{M^2 \mp N^2}{2MN},$$

whence, by addition, $m/n = M/N$. Since these are irreducible fractions with positive terms, we get $m = M$, $n = N$. Then $l = L$.

4. **Rational oblique triangles.** A triangle is called rational if its sides and area are expressed by rational numbers. Let ABC be a rational triangle whose sides are designated by $a = BC$, $b = AC$, $c = AB$. Since



$$a^2 = b^2 + c^2 - 2bc \cos A,$$

$\cos A$ is rational. Hence AF is rational. Since the area equals $\frac{1}{2}BF \cdot AC$ and is rational, BF also is rational. Hence *every rational oblique triangle may be formed by juxtaposing two rational right triangles.*

By § 2, the sides of any rational right triangle are proportional to

$$2, \quad \frac{m^2 - n^2}{mn}, \quad \frac{m^2 + n^2}{mn};$$

and the sides of any second rational right triangle are proportional to

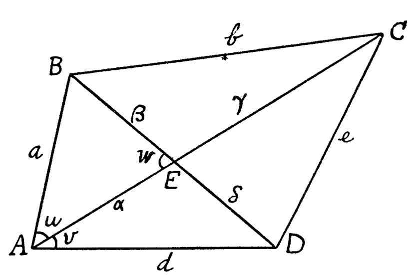
$$2, \quad \frac{M^2 - N^2}{MN}, \quad \frac{M^2 + N^2}{MN}.$$

Juxtaposing these right triangles, we see that the sides of any oblique rational triangle are proportional to

$$(3) \quad \frac{m^2 + n^2}{mn}, \quad \frac{M^2 + N^2}{MN}, \quad \frac{m^2 - n^2}{mn} \pm \frac{M^2 - N^2}{MN} = \frac{(mN \pm nM)(mM \mp nN)}{mnMN},$$

with the upper or lower signs according as the component right triangles do not or do overlap. This result was stated by Euler[1] in a posthumous fragment, but the portion of his paper which contained his proof is missing.

5. **Quadrilaterals with rational sides and diagonals.** Following Kummer,[2] we first prove that *the segments of the diagonals are rational.* By



$$b^2 = a^2 + AC^2 - 2a \cdot AC \cdot \cos u,$$

$\cos u$ is rational. Similarly, $\cos v$ and $\cos (u + v)$ are rational. Hence $\sin u \cdot \sin v$ and $\sin^2 u = 1 - \cos^2 u$ are rational. By division,

[1] *Comm. Arith. Coll.*, vol. 2, 1849, p. 648; *Opera Postuma*, vol. 1, 1862, p. 101.
[2] *Journal für reine und angewandte Mathematik*, vol. 37, 1848, pp. 1–20.

we see that $\sin u/\sin v$ is rational. Then by

$$\frac{a}{\beta}=\frac{\sin w}{\sin u},\qquad \frac{d}{\delta}=\frac{\sin w}{\sin v},\qquad \frac{\beta}{\delta}=\frac{a}{d}\cdot\frac{\sin u}{\sin v},$$

β/δ is rational. Thus $1+\beta/\delta = BD/\delta$ is rational. Thus δ and β are rational. Similarly, the segments α and γ of the other diagonal are rational.

The next step is much simpler than that by Kummer, who separated two cases and defined ξ algebraically, but not trigonometrically as here. *The ratios of the sides of any triangle ABE are rational if and only if*

$$c=\cos w,\qquad \xi=\frac{\sin w}{\sin u}(1+\cos u) \tag{4}$$

are rational. These are necessary conditions for the rationality of the ratios of the sides in view of the law of cosines and $\sin w/\sin u = a/\beta$. They are sufficient conditions since

$$\frac{1}{\xi}=\frac{1-\cos u}{\sin w\sin u},\qquad \xi+\frac{\sin^2 w}{\xi}=\frac{2\sin w}{\sin u}=\frac{2a}{\beta},\qquad \xi-\frac{\sin^2 w}{\xi}=\frac{2\sin w}{\sin u}\cdot\cos u,$$

$$\alpha=a\cos u+\beta\cos w,\qquad \frac{\alpha}{\beta}=\frac{a}{\beta}\cos u+c,$$

so that

$$\frac{a}{\beta}=\frac{1}{2}\left(\xi+\frac{1-c^2}{\xi}\right),\qquad \frac{\alpha}{\beta}=\frac{1}{2}\left(\xi-\frac{\sin^2 w}{\xi}\right)+c=\frac{(\xi+c)^2-1}{2\xi}.$$

Applying this result to the four triangles with the common vertex E, we see that there must be rational numbers ξ, η, x, y, c such that $|c|<1$ and

$$\frac{\alpha}{\beta}=\frac{(\xi+c)^2-1}{2\xi},\qquad \frac{\gamma}{\beta}=\frac{(\eta-c)^2-1}{2\eta},\qquad \frac{\delta}{\alpha}=\frac{(x-c)^2-1}{2x},\qquad \frac{\delta}{\gamma}=\frac{(y+c)^2-1}{2y}. \tag{5}$$

The product of the first and third or second and fourth left members is δ/β. Hence must

$$\frac{(\xi+c)^2-1}{2\xi}\cdot\frac{(x-c)^2-1}{2x}=\frac{(\eta-c)^2-1}{2\eta}\cdot\frac{(y+c)^2-1}{2y}. \tag{6}$$

Hence for any set of rational solutions ξ, η, x, y, c of (6) for which $|c|<1$, we obtain a quadrilateral the ratios of whose sides and diagonals are all rational, since (5) are then consistent and give rational ratios for $\alpha, \beta, \gamma, \delta$, while, as shown above,

$$\frac{a}{\beta}=\frac{\xi^2+t}{2\xi},\qquad \frac{b}{\beta}=\frac{\eta^2+t}{2\eta},\qquad \frac{e}{\gamma}=\frac{y^2+t}{2y},\qquad \frac{d}{\alpha}=\frac{x^2+t}{2x},$$

where $t=1-c^2$. Evidently we may take $\beta=1$. Thus we may assign any rational values to ξ, η, c, with $|c|<1$, and seek the rational values of x for

which the quadratic equation (6) for y has rational roots, i.e., for which its discriminant

$$\{\alpha x^2 - 2c(\alpha + \gamma)x - \alpha t\}^2 + 4t\gamma^2x^2 \tag{7}$$

is a rational square. Only tentative methods are known for making such a quartic function of x equal to a rational square. From one such value of x others may be found by Euler's method.[1] Starting from simple values found by inspection, Kummer deduced various special rules for forming quadrilaterals with rational sides and diagonals.

6. **Rational quadrilaterals.** A quadrilateral is called rational if its sides, diagonals and area are all expressed by rational numbers. The area of $ABCD$ is

$$\tfrac{1}{2}(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha) \sin w.$$

Hence to the conditions in § 5 we must annex the condition that also $\sin w$ be rational (Kummer, *loc. cit.*). The rational solutions of $\sin^2 w + c^2 = 1$ are (§ 2)

$$\sin w = \frac{2\lambda}{\lambda^2 + 1}, \qquad c = \frac{\lambda^2 - 1}{\lambda^2 + 1},$$

where λ is rational. Hence to find all rational quadrilaterals we proceed as in § 5, but restrict c to numbers of the form $(\lambda^2 - 1)/(\lambda^2 + 1)$. For instance, we may assign arbitrary rational values to ξ, η, λ, and seek the rational numbers x for which the quartic function (7) is a rational square.

7. **Quadrilaterals formed by juxtaposing four triangles.** The fact that every rational oblique triangle may be formed by juxtaposing two rational right triangles (§ 4) suggests that a similar attempt be made for quadrilaterals. Waiving the requirement of rational area, we seek quadrilaterals whose sides and diagonals shall be rational and hence (§ 5) also the segments of the diagonals. As the first component triangle AEB take one whose sides a, α, β are measured by any rational numbers satisfying the necessary inequalities. Then

$$a^2 = \alpha^2 + \beta^2 - 2\alpha\beta c, \qquad c = \cos w, \tag{8}$$

determines c rationally. In the second component triangle BEC, we have given side β and angle $180° - w$, and seek rational values of γ and b such that

$$b^2 = \beta^2 + \gamma^2 + 2\beta\gamma c. \tag{9}$$

In view of (8), we know the solution $b = a$, $\gamma = -\alpha$. Hence we set

$$\gamma = -\alpha + z, \qquad b = a + kz, \qquad z \neq 0,$$

where z and k are to be found rationally. Then (9) reduces by means of (8) to $(k^2 - 1)z = 2\beta c - 2\alpha - 2ak$. If $k^2 = 1$, either $a + z = b$ or $b + z = a$, whereas a, b, and $z = \alpha + \gamma = AC$ are sides of a triangle. Hence

$$z = \frac{2\beta c - 2\alpha - 2ak}{k^2 - 1}, \tag{10}$$

[1] L. E. Dickson, *History of the Theory of Numbers*, vol. 2, 1920, p. 639 *seq.*

where k is an arbitrary rational number distinct from ± 1. In the third component triangle CED, we have given side γ and angle w, and seek rational solutions e and δ of

$$e^2 = \gamma^2 + \delta^2 - 2\gamma\delta c.$$

For $\delta = -\beta + x$, $e = b + lx$, this reduces by means of (9) to

$$x = \frac{-2\gamma c - 2\beta - 2bl}{l^2 - 1}. \tag{11}$$

The fourth component triangle AED has two given sides α and δ and the included angle $180° - w$. The condition for rational closure is that $\alpha^2 + \delta^2 + 2\alpha\delta c$ be the square of a rational number d. Replacing δ by $x - \beta$, we get

$$X^2 + t\alpha^2 = d^2, \qquad X = x + \alpha c - \beta, \qquad t = 1 - c^2.$$

The complete solution in rational numbers, found by writing $d = X + m$, is

$$X = \frac{t\alpha^2 - m^2}{2m}, \qquad d = \frac{t\alpha^2 + m^2}{2m},$$

where m is a rational number $\neq 0$. Comparing the resulting value of x with its former value (11), we obtain a single condition on our rational parameters a, α, β, k, l, m, in terms of which all the remaining quantities are expressible rationally. While the present method is more natural than Kummer's and explains intuitively why any method must involve a rational condition of closure [(6) in Kummer's method], his method has the advantage of the symmetry due to his simultaneous consideration of the four component triangles.

8. **Parallelograms with rational sides and diagonals.** The component triangles are equal in pairs. To apply Kummer's method we need consider only the first two fractions (5), which must be equal since $\gamma = \alpha$. Hence shall

$$f \equiv \xi - \frac{t}{\xi} + 4c = \eta - \frac{t}{\eta}, \qquad \eta^2 - f\eta - t = 0,$$

$$\xi^2(2\eta - f)^2 = \xi^2 f^2 + 4t\xi^2 = (\xi^2 + 4c\xi - t)^2 + 4t\xi^2.$$

This final sum, which is therefore to be a rational square, may be deduced directly from (7) by removing the factor $\alpha^2 = \gamma^2$, and writing ξ for $-x$.

We may avoid this difficult problem of making a quartic function equal to a rational square by proceeding as in § 7, where we now require only equations (8) and (9). As before, (8) merely serves to determine c. When this value of c is inserted into (9) with $\gamma = \alpha$, we obtain the same result as if we added (8) to (9):

$$a^2 + b^2 = 2(\alpha^2 + \beta^2). \tag{12}$$

Set $\alpha + \beta = h$, $\alpha - \beta = g$. Then (12) becomes

$$a^2 + b^2 = g^2 + h^2. \tag{13}$$

To solve this in integers, set $a + g = mq$, $h + b = nq$, where m and n are relatively prime. Then $a - g = np$, $h - b = mp$. Thus

$$(14)\quad a = \tfrac{1}{2}(mq + np),\quad g = \tfrac{1}{2}(mq - np),\quad h = \tfrac{1}{2}(nq + mp),\quad b = \tfrac{1}{2}(nq - mp).$$

Since m and n are relative prime, these four numbers are integers only in the following cases: mn odd, p and q both even or both odd; just one of m and n even, p and q even. *All integral solutions of* (13) *are given by* (14) *in which* m *and* n *are relatively prime integers, while* p *and* q *are both even or* p, q, m, n *are all odd.*

The last problem is evidently the same as that of finding all triangles whose three sides and one median are rational.

9. **Conclusion.** All of the problems mentioned in this paper have been completely solved except that of a general quadrilateral whose sides and diagonals (and area) are rational. That question reduces to the problem of making a quartic function equal to a rational square. To this same problem may be reduced the solution of various questions[1] relating to triangles and quadrilaterals, as well as many questions in Diophantine analysis. A complete solution of this common outstanding problem is much to be desired.

THE TRIANGLE OF REFERENCE IN ELEMENTARY ANALYTIC GEOMETRY.

By LENNIE PHOEBE COPELAND, Wellesley College.

While the use of the triangle of reference and homogeneous coördinates is common in advanced work in mathematics, comparatively little is done along this line by undergraduates. It seems possible that younger students might find it profitable and interesting to note the behavior and shape, especially at infinity, of some of the well-known curves, when plotted on a triangle of reference. This triangle may be explained very simply,[2] since it is formed by the three lines of reference (Fig. 1) CA, CB and AB or $y = 0$, $x = 0$ and $z = 0$ corresponding respectively to the X and Y axes of the Cartesian system and the "conventional line at infinity." The distances from these lines, numbers proportional to them, or to arbitrary multiples of them, determine the coördinates of any point. The selection of the negative and positive sides of the lines may be made arbitrarily. However, it is generally more convenient to determine them in such a manner that the coördinates of points within the triangle shall be positive. This region

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[1] Dickson, *History of the Theory of Numbers*, vol. 2, 1920, pp. 165–224, 497.

[2] G. Salmon, *Treatise on Conic Sections*, fifth edition, London, 1869, chapter 4.

G. Salmon, *Higher Plane Curves*, second edition, Dublin, 1873, chapter 1.

C. A. Scott, *Introductory Account of Certain Modern Ideas and Methods in Plane Analytical Geometry*, London, 1894.

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O. Veblen and J. W. Young, *Projective Geometry*, vol. 1, Boston, 1910, p. 174.